Resolution of Brinkman Equations with a New Boundary Condition by Using Mixed Finite Element Method

Omar El Moutea, Hassan El Amri and Abdeslam Elakkad

Abstract This paper considers numerical methods for solving Brinkman equations with a new boundary condition summing Dirichlet and Neumann conditions. We develop here a robust stabilized mixed finite element method (MFEM), and two types of a posteriori error indicator are introduced to give global error estimates; there are equivalent to the true error. We present numerical simulations.

Keywords Brinkman equations · C_{a,μ^*} boundary condition · Mixed finite element methods · Residual error estimator · Numerical simulations

1 Introduction

This work deals with the development of stable numerical methods for the Brinkman equations; these equations are very important in a different domain, for example: in hydrogeology, porous media, and petroleum engineering. By these equations, we can model the flow in complex situations, for example, coupling flow in porous media and surface flow of fluids and we use these equations if we have different domains with variable coefficients. To describe the flow of a viscous fluid, see [1] and in soil mechanics see [2, 3]. Mathematically, this equation is a combination of two partial differential equations; we use a new boundary condition (generalizes the Dirichlet and the Neumann conditions). This boundary condition is used for Stokes

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V. Bhateja et al. (eds.), *Embedded Systems and Artificial Intelligence*, Advances in Intelligent Systems and Computing 1076,
https://doi.org/10.1007/978-981-15-0947-6_23

problem in [4, 5]. The weak formulation of this equation is a problem of saddle point type which is our case in this study to show the existence, the uniqueness of the solution of this problem see [6, 7]. During the last decades, a posteriori error analysis in problems related to fluid dynamics is a very important subject that has received a lot of attention. For the conforming case, there are different ways to define error estimators by using the residual equation. In particular, for the Stokes problem, Ainsworth and Oden [8], Bank and Welfert [8], and Verfurth [9] introduced several error estimators and show that they are equivalent to the energy norm of the errors.

The plan of the paper is as follows. In Sect. 2, we present the model problem used in this paper. The weak formulation of our problem is presented, and we show the existence and uniqueness of the solution in Sect. 3. The discretization by classical mixed finite elements is described in Sect. 4. In Sect. 5, we perform the same analysis for this introduced two types of a posteriori error bounds of the computed solution. We present a numerical test in Sect. 6.

2 Governing Equations

Let Ω be an open bounded polygonal or polyhedral reservoir in $\mathbb{R}^2$ and $\Gamma = \partial\Omega$ its boundary. The simplest form of Brinkman's equation is to search the unknowns velocity functions and pressure of the fluid satisfying

$$\begin{cases} -\mu^*\nabla^2\overrightarrow{u} + \mu K^{-1}\mathbf{u} + \nabla p = \overrightarrow{f} & \textbf{in } \Omega \\ \nabla\cdot\overrightarrow{u} = 0 & \textbf{in } \Omega. \end{cases} \tag{1}$$

The function $\overrightarrow{f}$ is a momentum source term, μ denotes the fluid viscosity, μ^* the effective viscosity of the fluid, and K is the permeability tensor of the porous media, which may contain multiscale features of the media.

We assume that these functions $\mu,\ \mu^* \in L^\infty(\Omega)$, and the tensor K are symmetric definite positive, which is uniformly elliptic; i.e., there exist two positive constants $\gamma_{\min}, \gamma_{\max}$ such that

$$\gamma_{\min}\,|\eta|^2 \le \eta^t K\eta \le \gamma_{\max}\,|\eta|^2 \tag{2}$$

for all $\eta \in L^2(\Omega)$ and $x \in \Omega$.

The problem consists of finding a velocity $\overrightarrow{u}$ and a pressure p fields with the C_{a,μ^*} boundary condition defined by

$$a\overrightarrow{u} + (\mu^*\nabla\overrightarrow{u} - pI)\overrightarrow{n} = \overrightarrow{t} \quad \textbf{in } \Gamma = \partial\Omega. \tag{3}$$

We will consider the fluid viscosity and the effective viscosity of the fluid are bounded functions depend on the spaces. In the boundary condition (3), the functions $\overrightarrow{t}$, a and μ^* are bounded polynomials such that

$$\alpha_1 \leq \frac{a(x)}{\mu^*(x)} \leq \beta_1 \ \forall x \in \Gamma \tag{4}$$

where the constants $\alpha_1 \in \mathbb{R}^+$ and $\beta_1 \in \mathbb{R}^+$.

Remark 1 Let the functions a and μ^* (two nonzeros defined on $\partial\Omega$ are strictly positive constants), if $a << \mu^*$ then C_{a,μ^*} is the Neumann boundary condition, else if $\mu^* << a$ then C_{a,μ^*} is the Dirichlet boundary condition, for that the boundary condition C_{a,μ^*} generalized Dirichlet–Neumann conditions.

3 Weak Formulation and Existence and Uniqueness of the Solution

Before starting to define the weak formulation of our problem, we define different spaces used in this study see [10]. For more details on the notation or spaces used in this part, see [4, 5].

3.1 The Weak Formulation

The variational formulation of (1)–(3) reads, find $\overrightarrow{u} \in H^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that:

$$\begin{cases} \int_\Omega \mu^* \nabla \overrightarrow{u} \cdot \nabla \overrightarrow{v} + \int_\Gamma a \overrightarrow{u} \cdot \overrightarrow{v} + \int_\Omega \mu K^{-1} \overrightarrow{u} \cdot \overrightarrow{v} - \int_\Omega p \nabla \cdot \overrightarrow{v} \\ \quad = \int_\Omega \overrightarrow{f} \cdot \overrightarrow{v} + \int_\Gamma \overrightarrow{t} \cdot \overrightarrow{v} \\ \int_\Omega q \nabla \cdot \overrightarrow{u} = 0 \end{cases} \tag{5}$$

for all $\overrightarrow{v} \in H^1(\Omega)$ and $q \in L_0^2(\Omega)$.

To simplify this study, we use these notations

$$a(\overrightarrow{u}, \overrightarrow{v}) = \int_\Omega \mu^* \nabla \overrightarrow{u} \cdot \nabla \overrightarrow{v} + \int_\Gamma a \overrightarrow{u} \cdot \overrightarrow{v} + \int_\Omega \mu K^{-1} \overrightarrow{u} \cdot \overrightarrow{v},$$
$$L(\overrightarrow{v}) = \int_\Omega \overrightarrow{f} \cdot \overrightarrow{v} + \int_\Gamma \overrightarrow{t} \cdot \overrightarrow{v}$$

and

$$b(\overrightarrow{u}, q) = \int_\Omega q \nabla \cdot \overrightarrow{u}.$$

The system is written to find $\overrightarrow{u} \in H^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that:

$$\begin{cases} a(\overrightarrow{u}, \overrightarrow{v}) + b(\overrightarrow{v}, p) = L(\overrightarrow{v}) \\ b(\overrightarrow{u}, q) = 0 \end{cases} \tag{6}$$

for all $\overrightarrow{v} \in H^1(\Omega)$ and $q \in L_0^2(\Omega)$.

3.2 The Existence and Uniqueness of the Solution

In this part, we will study the existence and uniqueness of the variational formulation of our problem (6), for that, we recall important inequalities, which will be used in this analysis. Firstly, we can see that the space $(H^1(\Omega), \|\overrightarrow{v}\|_{J,\Omega})$ is a Hilbert space, which is obliged condition in the existence and uniqueness of the solution, for that we need the following results:

Theorem 1 *There exists two positives constants c_1 and c_2 such that:*

$$c_1\|\overrightarrow{v}\|_{1,\Omega} \leq \|\overrightarrow{v}\|_{J,\Omega} \leq c_2\|\overrightarrow{v}\|_{1,\Omega} \tag{7}$$

for all $\overrightarrow{v} \in H^1(\Omega)$.

Proof See [4, 5].

Theorem 2 *The space $(H^1(\Omega), \|\cdot\|_{J,\Omega})$ is a real Hilbert space.*

Proof $(H^1(\Omega), \|\cdot\|_{1,\Omega})$ is a real Hilbert space, the norms $\|\cdot\|_{1,\Omega}$ and $\|\cdot\|_{J,\Omega}$ are equivalent, then $(H^1(\Omega), \|\cdot\|_{J,\Omega})$ is a real Hilbert space.

We can see, now, the existence and uniqueness of the solution.

Theorem 3 *The function $b(\cdot,\cdot)$ is satisfies the velocity–pressure $inf - sup$ condition, there exists a constant $\beta > 0$ such that:*

$$\sup_{\overrightarrow{v} \in H^1(\Omega)} \frac{b(\overrightarrow{v}, q)}{\|\overrightarrow{v}\|_{J,\Omega}} \geq \beta\|q\|_{0,\Omega} \tag{8}$$

for all $q \in L_0^2(\Omega)$.

Proof The same proof of [7] suffices to see that $H_0^1(\Omega) \subset H^1(\Omega)$ and $\|\overrightarrow{v}\|_{J,\Omega} = |\overrightarrow{v}|_{1,\Omega}$ in $H_0^1(\Omega)$.

Using "big" symmetric bilinear form $C[(\overrightarrow{u}, p), (\overrightarrow{v}, q)]$ and the corresponding function $F(\overrightarrow{v}, q)$. The bilinear form $a(\cdot,\cdot)$ is positive continuous $H^1(\Omega) - elliptic$ and the bilinear form $b(\cdot,\cdot)$ is continuous satisfies the $inf - sup$ condition. Now, we present a stabilized finite element scheme for the Brinkman problem.

Find $\overrightarrow{u} \in H^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that:

$$C[(\overrightarrow{u}, p), (\overrightarrow{v}, q)] = F(\overrightarrow{v}, q) \tag{9}$$

for all $\overrightarrow{v} \in H^1(\Omega)$ and $q \in L_0^2(\Omega)$. Then, the problem (6) is well-posed, and the form bilinear C satisfies the following propositions.

Proposition 1 *For all* $(\overrightarrow{w}, s) \in H^1(\Omega) \times L_0^2(\Omega)$, *we have*

$$\sup_{(\overrightarrow{v}, q) \in H^1 \times L_0^2} \frac{C[(\overrightarrow{u}, p), (\overrightarrow{v}, q)]}{\|\overrightarrow{v}\|_{J,\Omega} + \|q\|_{0,\Omega}} \geq \delta(\|\overrightarrow{w}\|_{J,\Omega} + \|s\|_{0,\Omega}). \tag{10}$$

Proof See [10].

4 Mixed Finite Element Approximation

In this section, we use the finite element method to solve this problem see [11]. We consider the family of triangulations T_h, of our domain Ω where $h > 0$. For any triangle $T \in T_h$ and for an element edge E, we define these notations

- ω_T is of triangle sharing at least one edge with element T,
- $\widetilde{\omega}_T$ is the set of triangles sharing at least one vertex with T,
- ∂T is the set of the four edges of T, $\varepsilon(T)$ the set of its edges and N_T vertices.
- ω_E denotes the union of triangles sharing E,
- $\widetilde{\omega}_E$ is the set of triangles sharing at least one vertex whit E.

We let $\varepsilon_h = \bigcup_{T \in T_k} \varepsilon(T)$ denotes the set of all edges split into interior and boundary edges $\varepsilon_h = \varepsilon_{h,\Omega} \bigcup \varepsilon_{h,\Gamma}$ where

$$\varepsilon_{h,\Omega} = \{E \in \varepsilon_h : E \subset \Omega\} \textbf{ and } \varepsilon_{h,\Gamma} = \{E \in \varepsilon_h : E \subset \partial\Omega\}.$$

We denote by h_T the diameter of a simplex, h_E the diameter of a face E of T and $h = \max_{T \in T_k}\{h_T\}$. We define FE spaces $X_h^1 \subset H^1(\Omega)$ and $M^h \subset L_0^2(\Omega)$. The discrete version of (6) is, find $\overrightarrow{u}_h \in X_h^1$ and $p_h \in M^h$ such that:

$$\begin{cases} a(\overrightarrow{u}_h, \overrightarrow{v}_h) + b(\overrightarrow{v_h}, p_h) = L(\overrightarrow{v}_h), \\ b(\overrightarrow{u}_h, q_h) = 0, \end{cases} \tag{11}$$

for all $\overrightarrow{v}_h \in X_h^1$ and $q_h \in M_h$.

Note that, all the results remain valid for these spaces X_h^1 and M^h.

5 A Posteriori Error Estimator

In this section, we use two types of a posteriori error indicator: the first, residual error estimator and, the second, local Poisson problem estimator. These errors give global error estimates where there are equivalent to the true error. For the a posteriori error estimation for stabilized mixed approximations of the Stokes equations see [12].

5.1 A Residual Error Estimator

In this paper, we use MINI element method; they use a function called the "bubble function," which is related to any element of the space meshing. In Ceruse et al. ([13], Lemma 4.1), we established the Clement interpolation estimate.

Our aim is to estimate the velocity error $\overrightarrow{u} - \overrightarrow{u}_h \in H^1(\Omega)$ and the pressure error $p - p_h \in L_0^2(\Omega)$. The element of contribution $\eta_{R,T}$ is given by

$$\eta_{R,T}^2 = h_T^2 \|\overrightarrow{R}_T\|_{0,T}^2 + \|R_T\|_{0,T}^2 + \sum_{E \in \partial T} h_E \|\overrightarrow{R}_E\|_{0,E}^2, \tag{12}$$

the components of the residual error estimator $\overrightarrow{R}_T$, R_T in (12) are given by

$$\overrightarrow{R}_T = \{\overrightarrow{f} + \mu^* \nabla^2 \overrightarrow{u}_h - \mu K^{-1} \overrightarrow{u}_h - \nabla p_h\}|_T \tag{13}$$

and

$$R_T = \{\nabla \cdot \overrightarrow{u}_h\}|_T. \tag{14}$$

The residual error estimator $\mathbf{R}_E$ is given by

$$\mathbf{R}_E = \begin{cases} \frac{1}{2}[|\mu^* \nabla \mathbf{u}_h - p_h I|] \text{ if } E \in \varepsilon_{h,\Omega} \\ \overrightarrow{t} - [a \overrightarrow{u}_h + (\mu^* \nabla \overrightarrow{u}_h - p_h I)\overrightarrow{n}] \text{ if } E \in \varepsilon_{h,\Gamma}. \end{cases} \tag{15}$$

With the key contribution coming from the stress jump associated with an edge E adjoining elements T and S:

$$[[\mu^* \nabla \overrightarrow{u}_h - p_h I]] = ((\mu^* \nabla \overrightarrow{u_h} - p_h I)|_T - (\mu^* \nabla \overrightarrow{u}_h - p_h I)|_S)\overrightarrow{n}_{E,T}. \tag{16}$$

The global residual error estimator is given by:

$$\eta_R = \left(\sum_{T \in \tau_h} \eta_{R,T}^2 \right)^{\frac{1}{2}}. \tag{17}$$

Our aim is to bound $\|\overrightarrow{u} - \overrightarrow{u}_h\|_X$ and $\|p - p_h\|_M$ with respect to the norm $\|\cdot\|_J$ for the quotient velocity norm $\|\overrightarrow{v}\|_X = \|\overrightarrow{v}\|_{J,\Omega}$ and the pressure norm $\|p\|_X = \|p\|_{0,\Omega}$. For any $T \in T_h$ and $E \in \partial T$, we define the following functions:

$$\overrightarrow{w}_T = \overrightarrow{R}_T b_T,\ \overrightarrow{w}_E = \overrightarrow{R}_E b_E$$

where this functions verified
• $\overrightarrow{w}_T = \overrightarrow{0}$ on ∂T.
• if $E \in \partial T \cap \varepsilon_{h,\Omega}$
then $\overrightarrow{w}_E = \overrightarrow{0}$ on $\partial \omega_E$,
• if $E \in \partial T \cap \varepsilon_{h,\Gamma}$ *then* $\overrightarrow{w}_E = \overrightarrow{0}$ in the other three edges of triangle T.
• $\overrightarrow{w}_T$ and $\overrightarrow{w}_E$ can be extended to whole of Ω by setting:
$\overrightarrow{w}_T = \overrightarrow{0}$ in $\Omega - \overline{T}$
$\overrightarrow{w}_E = \overrightarrow{0}$ in $\Omega - \overline{\omega_E}$ if $E \in \partial T \cap \varepsilon_{h,\Omega}$.
$\overrightarrow{w}_T = \overrightarrow{0}$ in $\Omega - \overline{T}$ if $E \in \partial T \cap \varepsilon_{h,\Gamma}$.

With these two functions, we have the following lemmas.

Lemma 1 *For any $T \in T_h$ we have:*

$$\int_T \overrightarrow{f} \cdot \overrightarrow{w}_T = \int_T (\mu^* \nabla \overrightarrow{u} - pI) \cdot \nabla \overrightarrow{w}_T + \int_T \mu K^{-1} \overrightarrow{u} \cdot \overrightarrow{w}_T. \quad (18)$$

for all $\overrightarrow{u}$, $\overrightarrow{w}_T \in X_h^1$.

Lemma 2 *(i) if $E \in \partial T \cap \varepsilon_{h,\Omega}$, we have:*

$$\int_{\omega_E} \left(\overrightarrow{f} - \mu K^{-1} \overrightarrow{u}\right) \cdot \overrightarrow{w}_E = \int_{\omega_E} (\mu^* \nabla \overrightarrow{u} - pI) \cdot \nabla \overrightarrow{w}_E. \quad (19)$$

(ii) if $E \in \partial T \cap \varepsilon_{h,\Gamma}$, we have:

$$\int_T \left(\overrightarrow{f} - \mu K^{-1} \overrightarrow{u}\right) \cdot \overrightarrow{w}_E = \int_T (\mu^* \nabla \overrightarrow{u} - pI) \cdot \nabla \overrightarrow{w}_E + \int_{\partial T} (a\overrightarrow{u} - \overrightarrow{t}) \cdot \overrightarrow{w}_E. \quad (20)$$

Theorem 4 *For any mixed finite element approximation defined on triangular grids T_h, the residual estimator η_R satisfies:*

$$\|\overrightarrow{e}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \le C_\Omega \eta_R \quad (21)$$

$$\eta_{R,T} \le C \left(\sum_{T' \in \omega_T} \left\{ \|\overrightarrow{e}\|_{J,T'}^2 + \|\varepsilon\|_{0,T'}^2 \right\} \right)^{\frac{1}{2}}. \quad (22)$$

Note that, the constant C in the local lower bound is independent of the domain.

5.2 A Local Poisson Problem Estimator

The local Poisson problem estimator:

$$\eta_P = \sqrt{\sum_{T \in T_h} \eta_{P,T}^2} \tag{23}$$

as follows

$$\eta_{P,T}^2 = \|\mathbf{e}_{P,T}\|_{J,T}^2 + \|\varepsilon_{P,T}\|_{0,T}^2 \tag{24}$$

Theorem 5 *The estimator $\eta_{P,T}$ is equivalent to the $\eta_{R,T}$ estimator:*

$$c\eta_{P,T} \le \eta_{R,T} \le C\eta_{P,T} \tag{25}$$

Theorem 6 *For any mixed finite element approximation defined on triangular grids T_h the estimator η_P satisfies:*

$$\|\mathbf{e}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \le C\eta_P \tag{26}$$

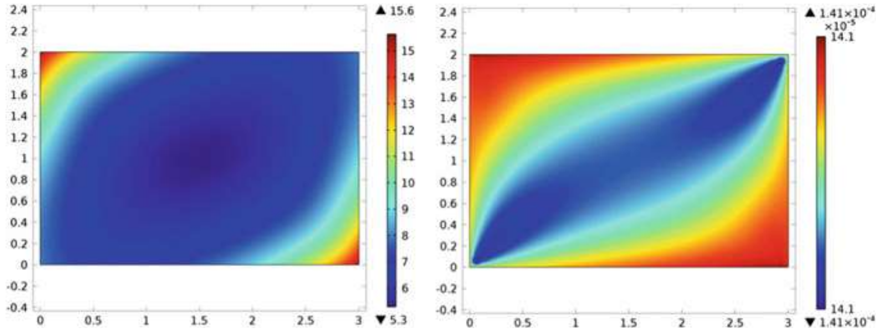
and

$$\eta_{P,T} \le C \left(\sum_{T \in T_h} \left\{ \|\mathbf{e}\|_{J,T'}^2 + \|\varepsilon\|_{0,T'}^2 \right\} \right)^{\frac{1}{2}} \tag{27}$$

The constant C in the local lower bound is independent of the domain.

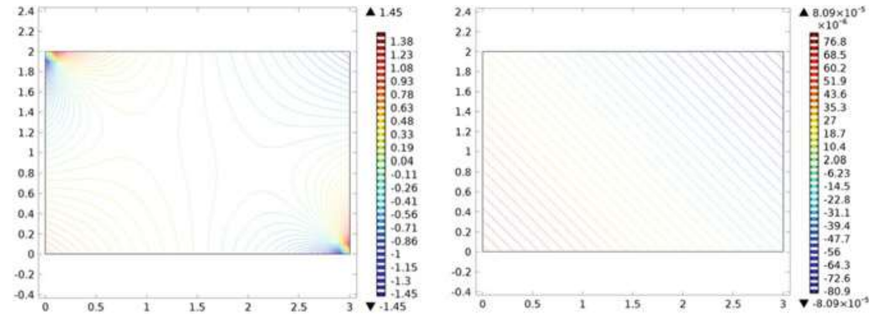
6 Numerical Simulation

In this section, we present numerical tests; based on the MFE method presented in this article, we use the simulator Comsol Multiphysics. The results obtained confirm that the errors are reasonable and the numerical computations of our problems have demonstrated that this approach yields a physically realistic flow. In this simulations, we take fluid density = 1000 kg/m^3, permeability = 0.3 m^2, Porosity = 0.4, and the fluid viscosity values ranging from 0.1 Pa $*$ s. We consider the boundary condition C_{a,μ^*}, where $\overrightarrow{t} = (1, 1)$, $a = 10^{-4}$ and $a = 10^3$. For discretization, we use uniform rectangular discretization of our reservoir. Figure 1 presents the velocity for $a = 10^{-4}$ and $a = 10^3$.



(a) The velocity with $a = 10^{-4}$ (b) The velocity with $a = 10^{3}$

Fig. 1 Velocity of Brinkman equation



(a) The pressure contours with $a = 10^{-4}$ (b) The pressure contours with $a = 10^{3}$

Fig. 2 Pressure contours of Brinkman equation

Table 1 Errors, the linear error, residual error, and the error of the velocity

h	Number of grids	erru	LinErr	LinRes
0.10	30×20	0.25	$1.6e^{-12}$	$8.0e^{-15}$
0.15	20×15	0.36	$1.8e^{-12}$	$1.0e^{-14}$
0.20	15×10	0.47	$2.4e^{-12}$	$1.9e^{-14}$
0.30	10×07	0.68	$3.8e^{-12}$	$2.7e^{-14}$
0.40	08×05	0.82	$5.5e^{-12}$	$5.0e^{-14}$

In Fig. 2, we show the pressure contours of Brinkman equation with different parameter in the boundary equation. Now, we present different errors for our problem. In Table 1, LinErr is the linear error, LinRes is the residual error, and erru is the error of the velocity equation. From Table 1, we can see the efficiency of this method, when the mesh is small enough the error approaches zero.

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